Recognizing isotropic electromagnetic medium from the behavior of phase velocity

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Abstract
It is known that in isotropic electromagnetic media wave propagation is determined by Lorentz geometry. In the setting of premetric non-dissipative media (where a medium is pointwise determined by 21 real parameters) we discuss the converse question: Is isotropic media the only class of media where wave propagation is determined by Lorentz geometry?

1 Introduction
Let us consider Maxwell’s equations on a 4-dimensional space $N$. In this setting, the electromagnetic medium is determined by a suitable tensor $\kappa$ on $N$, which pointwise depends on 36 real parameters. If $N = \mathbb{R} \times \mathbb{R}^3$, this corresponds to the usual Maxwell’s equations in $\mathbb{R}^3$ with the constitutive equation

$$
\begin{bmatrix}
H \\
D
\end{bmatrix} = \begin{bmatrix}
\mathcal{C} & \mathcal{B} \\
\mathcal{A} & \mathcal{D}
\end{bmatrix} \begin{bmatrix}
-E \\
B
\end{bmatrix},
$$

(1)

where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are $3 \times 3$ matrices with real coefficients; $-\mathcal{A}$ corresponds to permittivity, $\mathcal{B}^{-1}$ (when invertible) corresponds to permeability, and $\mathcal{C}, \mathcal{D}$ model possible magneto-electric effects. This formulation is known as the premetric formulation of electromagnetics. For a systematic presentation we refer to the book by Hehl and Obukhov [1]. We will also assume that the medium tensor $\kappa$ is non-dissipative. If we write the medium as in (1), this is equivalent to the symmetries $\mathcal{A} = \mathcal{A}^t$, $\mathcal{B} = \mathcal{B}^t$ and $\mathcal{C} = \mathcal{D}^t$ whence the medium pointwise depends on 21 real parameters. If $N = \mathbb{R} \times \mathbb{R}^3$ and the medium is time independent these symmetries imply that Poynting’s theorem holds. Lastly, we assume that the medium tensor $\kappa$ is invertible. Thus the state of the electromagnetic field is determined by fields $E$ and $B$ or, equivalently, by fields $H$ and $D$.

2 The Fresnel surface and phase velocity
By studying the behavior of propagating waves in the above setting, one can define the Fresnel surface $F(\kappa)$ for the medium tensor $\kappa$ [2]. In each point of space, the Fresnel surface $F(\kappa)$ is described by points $(\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^4$ that satisfy a 4th order polynomial equation

$$
\mathcal{G}^{ijkl} \xi_i \xi_j \xi_k \xi_l = 0.
$$

(2)

Here $\mathcal{G}^{ijkl}$ are coefficients for the Tamm-Rubilar tensor density that are determined by $\kappa$, and we have used the Einstein summing convention so that repeated indices are summed over 0, \ldots, 3. The Fresnel surface $F(\kappa)$ can be seen as an analogue of the dispersion equation, so that $F(\kappa)$ describes how phase velocity behaves in different directions. For example, in the special case that the medium tensor is isotropic, the Fresnel surface reduces to a Lorentz null cone $\left(\frac{1}{c^2} \xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2\right)^2 = 0$. Thus, in isotropic electromagnetic media, wave propagation is determined by
Lorentz geometry. However, for a general electromagnetic medium tensor $\kappa$ the Fresnel surface $F(\kappa)$ can have a complicated structure. If the medium is \textit{birefringent} (that is, differently polarized waves can propagate with different phase velocities) then $F(\kappa)$ can have multiple sheets with self-intersections. This is for example the case in uniaxial and biaxial crystals.

From the above we see that there are two different ways to describe an electromagnetic medium: One description is the medium tensor $\kappa$ that prescribes the coefficients in Maxwell’s equations. For example, this is the information required to solve the initial value problem for Maxwell’s equations. The second description is the Fresnel surface $F(\kappa)$. This description is more qualitative in that it only describes the dynamical response for propagating waves. If $\kappa$ is known we can compute the Fresnel surface $F(\kappa)$ by an explicit equation $[1, 2]$. A less well understood problem is the converse dependence.

\textbf{Question.} If $\kappa$ is unknown, then how much can one say about the anisotropic structure of $\kappa$ from $F(\kappa)$ alone?

A first observation is that $F(\kappa)$ can never uniquely determine $\kappa$. Known invariances for $F(\kappa)$ include invariance under scaling ($\kappa \mapsto C \kappa$ for $C \neq 0$), under addition of an axion component ($\kappa \mapsto \kappa + C \Id$ for $C \neq 0$), and under inversion ($\kappa \mapsto \kappa^{-1}$) when $\kappa$ is invertible.

A special case of the above question is to understand the class of medium tensors for which $F(\kappa)$ is a Lorentz null cone. That is, to understand the class of media where phase velocity behaves as in isotropic medium.

\textbf{Conjecture.} If $\kappa$ is non-dissipative and trace-free (that is, trace $\kappa = 0$) then $\kappa$ is isotropic if and only if the Fresnel surface is pointwise a Lorentz null cone.

The above conjecture was formulated (in a different but equivalent form) and studied in $[2, 3, 4, 5]$. See also the book $[1]$ by Hehl and Obukhov. Put differently, the conjecture states that the invariance $\kappa \mapsto \kappa + C \Id$ is the only invariance that prevents us from deciding if an unknown medium tensor is isotropic or not from phase velocity alone (in the class of non-dissipative premetric media).

The conjecture has been proven in a number of different settings: in the absence of magneto-electric effects, that is, for $\mathcal{C} = 0$, by Obukhov, Fukui and Rubilar [2], and in a special class of non-linear media by Obukhov and Rubilar [5]. On the level of the Tamm-Rubilar tensor density, Favaro and Bergamin have shown that if $g^{ijkl} \xi_i \xi_j \xi_k \xi_l = \sigma \left( g^{ij} \xi_i \xi_j \right)^2$ for a factor $\sigma$ and a Lorentz metric $g$, then the constitutive tensor $\kappa$ must be isotropic [6]. See also [6] for a discussion about the analogous problem for non-Lorentzian $g$. In [7] the conjecture was verified under the assumption that $\kappa$ is invertible.

The proof in [7] is an refinement of the argument in [8] which gives a classification of non-dissipative medium tensors for which the Fresnel surface is two Lorentz null cones. Let us note that even if the Fresnel surface of a medium tensor is one Lorentz null cone, the Fresnel equation could theoretically be of the form

$$(\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2)(\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2) = 0.$$  

(3)

In this case, the Fresnel surface (which only include real roots for $\xi_i$) would be the Lorentz null cone $\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0$. However, the medium would also support evanescent waves that correspond to complex roots to $\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 0$. This is why one needs to study factorisations like $g^{ijkl} \xi_i \xi_j \xi_k \xi_l = (A^{ij} \xi_i \xi_j)(B^{ij} \xi_i \xi_j)$ to understand medium tensors with one Lorentz null cone. An implications of [7] is that there exists no (non-dissipative and invertible) medium tensor that has (3) as a Fresnel equation. Let us also note that both [7] and [8] rely on the recent classification result for non-dissipative medium tensors by Schuller, Witte and Wohlfarth [9] and the computer algebra technique of Gröbner bases for simplifying systems of polynomial equations.
3 Outlook

All the above results assume that the medium tensor have real coefficients. This does exclude important classes of medium (like chiral medium) that can be described using complex coefficient medium and time harmonic fields. It is not clear how the above results would generalize to such a setting. For an initial discussion, see [7].

References


