A Least Squares Finite Element Formulation for Generalized Maxwell’s Boundary Value Problems in 2D

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Abstract

We formulate a finite element method for time harmonic generalized Maxwell’s equations in two dimensions. The formulation is based on a certain dual potential minimization problem in which the boundary data can be given as a tangential electric or tangential magnetic field. The formulation is verified with numerical tests.

1 Introduction

In this paper, we propose a finite element method for solving time harmonic electromagnetic boundary value problems based on generalized formulation of Maxwell’s equations [1, 2]. It turns out, that the slack variables introduced in the generalized Maxwell’s system of equations are identical to the slack variables appearing in first order system $LL^*$ (FOSLL*) finite element formulation of Maxwell’s equations [3, 4].

In the FOSLL* formalism, the variational formulation is laid down to a certain dual potentials [4]. In the case of Maxwell’s equations, the boundary conditions for electric or magnetic fields appear as linear functionals in the right hand side of the variational formulation. Intuitively, this is an expected result since the fields are given, in a sense, as derivatives of the dual potentials.

The rationale for considering the FOSLL* formalism together with generalized Maxwell’s equations is that the latter have very good static asymptotic properties [5] and the sesquilinear forms appearing in the FOSLL* methods are hermitian and positive definite by construction [3,4,6] which allows one to use conjugate gradient methods.

2 Generalized Maxwell system

We consider the time harmonic Maxwell’s equations for the $z$ component of normalized electric field $E$ and TM$_z$ normalized magnetic field $H$. They are given as follows (time signature $e^{-i\omega t}$, $\kappa = \omega\sqrt{\mu/\varepsilon}$):

\[
\begin{cases}
\nabla \cdot E = i\kappa H,
\n\nabla \times H = -i\kappa E + J \quad \text{in } \Omega,
\n\nabla \cdot H = 0
\end{cases}
\]

\[
\begin{cases}
E|_\Gamma = g \quad \text{or} \quad n \times H = f
\end{cases}
\quad \text{on } \Gamma.
\]

Here $\Omega$ is a Lipschitz domain in $\mathbb{R}^2$ and $\Gamma$ is its boundary. We shall call the boundary data to be of electric type if it is given by $E|_\Gamma$ and $n \times H$, and respectively, of magnetic type it is given by $n \times H$. The planar curl operators $\nabla \times$ and $\nabla \cdot$ are obtained by

\[
\begin{cases}
\nabla \cdot F(x,y) = \text{curl} \, u_z F(x,y) \quad \text{and}
\n\nabla \times F(x,y) = u_z \cdot \text{curl} \, F(x,y),
\end{cases}
\]

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where \textit{curl} is the usual three dimensional curl operator.

It should be noted that \( E \), \( \frac{\partial}{\partial n} \) and \( \mathbf{n} \cdot \mathbf{H} \) are related to each other through a local relation on the boundary \( \Gamma \) when \( \kappa \ne 0 \), and thus it is easy to check if \( g \) and \( f \) are consistent.

By introducing a slack variable \( \Phi \) to the Equation (1) one obtains an elliptic system given by

\[
\begin{pmatrix}
0 & -\mathbf{V} & 0 \\
\mathbf{V} & 0 & -\mathbf{V} \\
0 & -\mathbf{V} & 0
\end{pmatrix}
\begin{pmatrix}
E \\
\mathbf{H} \\
\Phi
\end{pmatrix}
= \begin{pmatrix}
-J \\
0 \\
0
\end{pmatrix}.
\]

(4)

We can make \( L \) unbounded skew-self adjoint on \( \mathcal{H} = (L_2)^4 \) if we take the domain \( \mathcal{D}(L) \) to be

\[
\mathcal{D}_E = H_0(\nabla^\perp) \times (H_0(\nabla) \cap H(\nabla \times)) \times H(\nabla)/\mathbb{C} \quad \text{or}
\]

\[
\mathcal{D}_M = (H(\nabla \times) \cap H_0(\nabla \times)) \times H_0(\nabla).
\]

(5)

(6)

Here \( \mathcal{D}_E \) and \( \mathcal{D}_M \) correspond to PEC and PMC boundaries, respectively.

The Sobolev space \( H(\nabla^\perp) \) consists of those \( L_2 \) functions \( f \) for which the norm \( \|f\|_{H(\nabla, \Omega)} := (\|f\|_{L_2}^2 + \|\nabla^\perp f\|_{L_2}^2)^{\frac{1}{2}} \) is finite. Here \( * \) is \( \frac{1}{2}, \cdot, \cdot \) or empty. The space \( H_0(\nabla^\perp) \) consists of those \( H(\nabla^\perp) \) functions whose normal or tangential trace vanishes if \( * \) is \( \nabla \) or \( \nabla \times \), respectively. It holds that \( H_0(\nabla^\perp) = H_0(\nabla) = H_0^1 \).

Formally, the FOSLL* method can be derived as follows [4]. Given a linear (possibly unbounded) operator \( L : \mathcal{H} \rightarrow \mathcal{H} \) with domain \( \mathcal{D}(L) \subset \mathcal{H} \) on a Hilbert space \( \mathcal{H} \) equipped with inner product \( \langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C} \), let us consider an abstract equation \( LU = F \). One tries to find its dual potential \( u \) as a minimizer of a functional \( J(v) = \|L^*v - U\|^2 \), where \( L^* \) is the Hilbert adjoint operator of \( L \). The minimizer of \( J \) satisfies \( L^*u = U \) if \( L^* \) is surjective.

By the usual variational argument, a minimizer \( u \) of \( J \) satisfies

\[
\langle L^*u, L^*v \rangle = \langle U, L^*v \rangle, \quad \forall v \in \mathcal{D}^*.
\]

(7)

Let us now denote \( U = [E \quad \mathbf{H} \quad \Phi]^T \in (L_2(\Omega))^4 \), \( F = [-J \quad 0 \quad 0]^T \) and \( v = [e \quad h \quad \phi]^T \).

If the boundary data is of electric kind, by integrating by parts the right hand side of the formula (7) one obtains

\[
\langle L^*u, L^*v \rangle = \begin{cases}
\langle F, v \rangle - \int_{\Gamma} E \mathbf{H} \cdot d\mathbf{A} - \int_{\Gamma} \mathbf{n} \cdot \mathbf{H} \phi \, d\sigma & \text{for } v \in \mathcal{D}_E \\
\langle F, v \rangle + \int_{\Gamma} \mathbf{n} \times \mathbf{H} d\sigma - \int_{\Gamma} \mathbf{n} \cdot \mathbf{H} \phi d\sigma & \text{for } v \in \mathcal{D}_M.
\end{cases}
\]

(8)

If \( U \) is a Maxwellian solution of (4), then \( \Phi = 0 \) [2], thus the last term in (8) vanishes.

3 Discretization and numerical tests

We use continuous piecewise \( k^0 \) order polynomial triangular elements to discretize (8). Let us denote such subspace by \( \mathcal{C}_{h,k} \) and its basis by \( \{\phi_i\} \). The parameter \( h \) corresponds to the maximum edge length of the mesh. We shall denote functions in \( \mathcal{C}_{h,k}^d \), \( d = 1, 2 \), by lower index \( h \). The vanishing normal or tangential trace of \( \mathbf{h}_h \) is obtained in weak form by Lagrange multipliers on the boundary. The use of multipliers makes sense since the discretized system (8) corresponds to a quadratic minimization problem in \( \mathcal{C}_{h,k}^d \) with linear constraints that \( \int \phi \mathbf{n} \cdot \mathbf{h}_h dA = 0 \) and \( \int \phi dA = 0 \) or \( \int \mathbf{n} \times \mathbf{h}_h dx = 0 \) and \( f_h dA = 0 \), where \( \phi \in \mathcal{C}_{h,k}(\Gamma) \).

The boundary data is taken to be that of a plane wave with \( \kappa = \pi/10 \) and \( u_k = \frac{1}{2} (\sqrt{3} u_x + u_y) \).

We study the convergence of the solution with respect to mesh parameter \( h \rightarrow 0 \) in a triangle domain (Fig. 1a) and in a corner domain (Fig. 1b).
We observe a convergence rate of $h^k$ (Figures 2a, 2b, 2c and 2d) for interior field. However, the boundary electric field exhibits slightly worse convergence rate for PEC boundary data when $k = 2$. In the corner domain, we find that no convergence is achieved (Fig. 2e). This is in line with earlier work [3] where weighted $L_2$ norms were used to overcome this problem in eddy current simulations.

4 Concluding remarks and future work

We introduced a finite element formulation for time harmonic two dimensional Maxwell’s boundary value problem in which the boundary data can be given as a tangential electric or magnetic field. By numerical tests in simple geometries we confirmed that the method converges in convex domains. In the corner domain, no convergence was achieved which is an expected result.

The sesquilinear forms in the variational equation for the dual potentials arise from an $L_2$ minimization problem and thus they are always hermitian and $\mathcal{D} \cap (H^1)^4$-elliptic which should imply that the arising stiffness matrices are hermitian and positive-definite [6]. However, since the discrete dual potentials are given in piecewise polynomial $(H^1)^4$ space one must ensure that the dual potentials have proper zero traces to the boundary. We implemented it by Lagrange multipliers, which in turn makes the stiffness matrix indefinite. A remedy for this would be to use constrained conjugate gradient methods [8] instead of Lagrange multipliers. Furthermore, in a square domain one can quite easily construct conformal finite element bases for $\mathcal{D}_E$ and $\mathcal{D}_M$.

It should be noted that in the FOSLL* method, it is easy to treat problems with non-smooth material data [3,4]. However, in this work we focused on the non-zero boundary value data instead of inhomogeneous material parameters.

The proposed method as such cannot be used to solve Maxwell’s equations in corner domains. This can be most likely overcome by weighted regularization methods [7] or any other method focused to mitigate the bad approximation behaviour of $H^1$ elements for $H_0(\nabla \cdot) \cap H(\nabla \times)$.

In the future work, we are interested in domain decomposition methods based on the variational formulations (8). It is also important to exploit the FOSLL* structure of the formulation by solving the discretized equation with conjugate gradients methods. This is particularly important in three dimensional problems where even small problems lead to discrete systems beyond the scope of direct solvers.

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Figure 2: Numerical convergence curves in triangle (a-d) and in corner domain. (e)

References


