Stochastic inversion in calculating the spectrum of signals with uneven sampling

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INTRODUCTION

The standard method of calculating the spectrum of a digital signal is based on the Fourier transform, which gives the amplitude and phase spectra at a set of equidistant frequencies from signal samples taken at equal intervals. In this paper a different method based on stochastic inversion is introduced. It does not imply a fixed sampling rate, and therefore it is useful in analysing geophysical signals which may be unequally sampled or may have missing data points. This could not be done by means of Fourier transform without preliminary interpolation. Another feature of the inversion method is that it allows unequal frequency steps in the spectrum, although this property is not needed in practice. The method has a close relation to methods based on least-squares fitting of sinusoidal functions to the signal. However, the number of frequency bins is not limited by the number of signal samples. In Fourier transform this can be achieved by means of additional zero-valued samples, but no such extra samples are used in this method. Finally, if the standard deviation of the samples is known, the method is also able to give error limits to the spectrum. This helps in recognising signal peaks in noisy spectra.

METHOD

Consider a model of a time signal $x(t)$ in terms of sinusoidal oscillations at m angular frequencies $\omega_k, k = 1, 2, \ldots, m$ with different amplitudes and phases, i.e.

$$x(t) = \sum_{k=1}^{m}(a_k \sin\omega_k t + b_k \cos\omega_k t). \tag{1}$$

By sampling the signal at times $t_j, j = 1, 2, \ldots, n$, we obtain a sample sequence of n numbers

$$x(t_j) = \sum_{k=1}^{m}(a_k \sin\omega_k t_j + b_k \cos\omega_k t_j) + \varepsilon_j, \quad j = 1, 2, \ldots, n. \tag{2}$$

Here sampling is considered as a measurement of the signal value and ε_j is the true measurement error (not an error estimate). Since $x(t)$ may contain a DC component, it is reasonable to choose $\omega_1 = 0$. Then (2) can be written as

$$x(t_j) = b_1 + \sum_{k=2}^{m}(a_k s_{jk} + b_k c_{jk}) + \varepsilon_j, \tag{3}$$

where

$$s_{jk} = \sin \omega_k t_j \text{ and } c_{jk} = \cos \omega_k t_j. \tag{4}$$

The amplitudes a_k and b_k are unknowns which are to be determined.

By collecting all unknonws, measurements and measurement errors into column vectors

$$\mathbf{X} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \\ a_2 \\ a_3 \\ \vdots \\ a_m \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x(t_1) \\ x(t_2) \\ \vdots \\ x(t_n) \end{pmatrix} \quad \text{and} \quad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}, \tag{5}$$

all equations of the form (3) can be written as a single matrix equation

$$\mathbf{x} = \bar{\mathbf{A}} \cdot \mathbf{X} + \boldsymbol{\varepsilon}, \tag{6}$$

where

$$\bar{\mathbf{A}} = \begin{pmatrix} 1 & c_{12} & c_{13} & \ldots & c_{1m} & s_{12} & s_{13} & \ldots & s_{1m} \\ 1 & c_{22} & c_{23} & \ldots & c_{2m} & s_{22} & s_{23} & \ldots & s_{2m} \\ 1 & c_{32} & c_{33} & \ldots & c_{3m} & s_{32} & s_{33} & \ldots & s_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c_{n2} & c_{n3} & \ldots & c_{nm} & s_{n2} & s_{n3} & \ldots & s_{nm} \end{pmatrix}. \tag{7}$$

This is a stochastic linear inversion problem with a well-known solution [1]

$$\mathbf{X}_0 = \bar{\mathbf{Q}}^{-1} \cdot (\bar{\mathbf{A}}^T \cdot \bar{\mathbf{\Sigma}}^{-1}) \cdot \mathbf{x}. \tag{8}$$

Here T indicates transpose,

$$\bar{\mathbf{\Sigma}} = \langle \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^T \rangle \tag{9}$$

is the covariance matrix of the Gaussian random error vector $\boldsymbol{\varepsilon}$ and

$$\bar{\mathbf{Q}} = \bar{\mathbf{A}}^T \cdot \bar{\mathbf{\Sigma}}^{-1} \cdot \bar{\mathbf{A}} \tag{10}$$

is the Fisher information matrix. The statistical errors of the unknowns are given by the posteriori covariance matrix

$$\bar{\mathbf{\Sigma}}_{\mathbf{X}} = \bar{\mathbf{Q}}^{-1}. \tag{11}$$

Solution (8) gives the unknonwn amplitudes a_k and b_k for each frequency component. When they are known, the power of the frequency component ω_k is given by

$$P_k = a_k^2 + b_k^2. \tag{12}$$

SAMPLE RESULTS

A special property of the method is that it allows the determination of a greater number of unknowns than the number of measurements. This means that the number of frequency bins in the spectrum can be large. However, the spectral resolution is determined by the length of the sample sequence in the same way as in Fourier transform. Adding more frequency bins has the same interpolating effect as adding zeros to the end of the sample sequence and using Fourier transform.

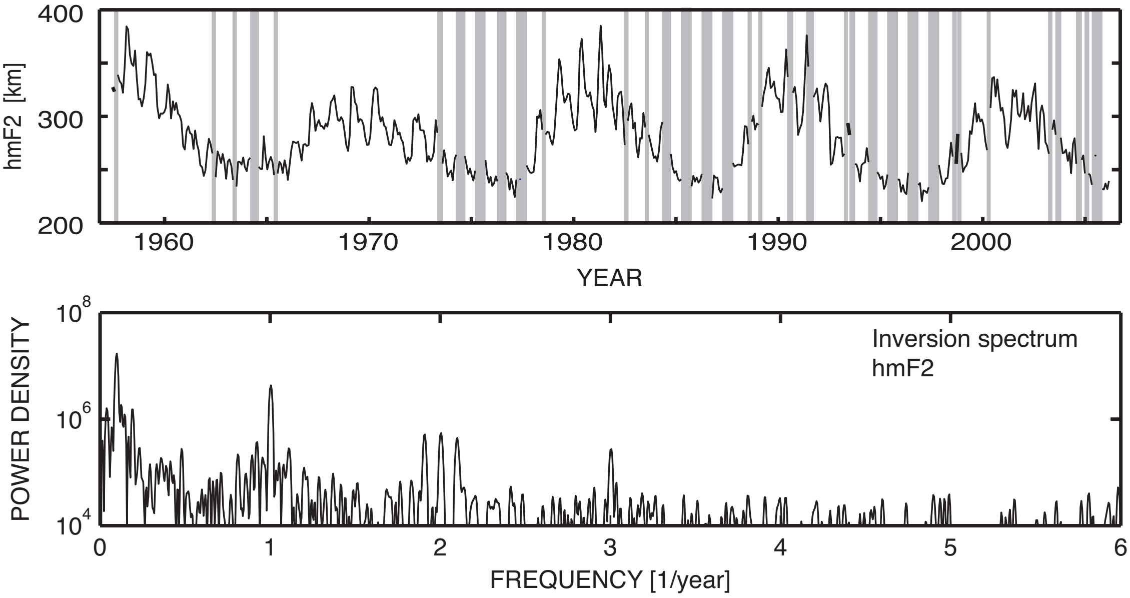

Figure 1: Example of a time series with gaps (grey stripes) and the calculated power spectrum.

As an example of the method, the power spectrum of the virtual height of the ionospheric F layer (hmF2) was calculated. The data were measured at Sodankylä, Finland, in 1957–2006, covering about 4.5 solar cycles. The peak altitude hmF2 was calculated from the peak plasma frequencies of the F2 and E layers and the maximum usable frequency for a 3000-km radio path [2]. Hourly monthly medians averaged over 10–14 LT were used in the analysis. Since months have different lengths, this leads to variable sampling intervals. The data also contains several gaps and therefore Fourier transform could not be directly applied to this time series.

The top panel of Fig. 1 shows the time series with intermediate gaps indicated by the grey stripes. The number of data points in this time series is 491. The power spectrum is shown in the bottom panel. The number of frequencies in the spectrum is 1200 (i.e. 2399 unknonwns in the inversion). The most prominent spectral peak at low frequencies is due to the solar cycle variation. The next peaks are the annual (1/year) and semiannual (2/year) variations and the second harmonic of the annual variation (3/year). An interesting feature is that, on both sides of the the semiannual variation, additional prominent spectral peaks appear. They are shifted from the semiannual peak by an amount which is equal to the frequency of the solar cycle variation. Hence they are due to nonlinear mixing of the semiannual variation with the solar cycle.

DISCUSSION

The method is only briefly introduced in this paper. A more thorough mathematical treatment shows that it is based on finding the most probable values of the amplitudes and phases of each frequency of the signal spectrum. This also explains why the number of unknowns can indeed exceed the number of measurement values. The single example shown in Fig. 1 demonstrates the power of the method in calculating spectra from data sequences with unequal sampling intervals and data gaps. Other examples have been calculated which indicate that even long gaps in the data have no effect on the widths of the spectral peaks but they only create random noise peaks. If a gap is too long, weak spectral peaks may be embedded in noise.

The method described in this paper is actually not completely new [3]. However, a new aspect in the present work is that the number of frequency bins in the spectrum is not limited by the number of samples as believed before. This is convincingly demonstrated by Fig. 1, as well as by a number of other examples. The use of stochastic inversion as a starting point also gives a deeper insight in the method than the previous approaches do. Furthermore, if error estimates of data samples are available, the method also gives a possibility to calculate an error limit for each spectral point. This, however, is not investigated here.

References

[1] T. Nygrén, M. Markkanen, M. Lehtinen, E.D. Tereshchenko, and B.Z: Khudukon, "Stochastic inversion in ionospheric radiotomography," *Radio Sci.*, vol. 32, pp. 2359–2372, 1997.

[2] P.A. Bradley, and J.R. Dudeney, "A simple model of the vertical distribution of electron concentration in the ionosphere," *J. Atmos. Terr. Phys.*, vol. 35, pp. 2131–2146, 1973.

[3] M.R. Craymer, "The least squares Spectrum, its inverse transform and autocorrelation function: Theory and some applications in geodesy," *(ftp://geod.nrcan.gc.ca/pub/GSD/craymer/pubs/thesis1998.pdf) Ph.D. Dissertation, University of Toronto, Canada*, 1998.