Electric images and image based potential for a layered conducting sphere

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INTRODUCTION

The geometry of the problem involves a homogeneous sphere of conductivity σ_b and radius b, covered by a concentric layer of conductivity σ_a and radius a. The ambient medium is nonconducting, $\sigma = 0$. A point dipole $\mathbf{Q}\delta(\mathbf{r}-\boldsymbol{\tau})$ inside the homogeneous sphere provokes a volume current, which can be observed from the electric field on the exterior surface $r = a$. The geometry models approximately the human head in EEG, for example. The BVP reads

$$\Delta u(\mathbf{r}) = 0, \qquad r > a, \tag{1}$$
$$\Delta u_a(\mathbf{r}) = 0, \qquad b < r < a, \tag{2}$$
$$\sigma_b \Delta u_b(\mathbf{r}) = \nabla \cdot \mathbf{Q}\delta(\mathbf{r}-\boldsymbol{\tau}), \qquad 0 \leq r < b. \tag{3}$$

with the interface conditions:

$$u(\mathbf{r}) = u_a(\mathbf{r}), \qquad \frac{\partial u_a(\mathbf{r})}{\partial r} = 0, \qquad r = a \tag{4}$$
$$u_a(\mathbf{r}) = u_b(\mathbf{r}), \qquad \sigma_a \frac{\partial u_a(\mathbf{r})}{\partial r} = \sigma_a \frac{\partial u_b(\mathbf{r})}{\partial r}, \qquad r = b, \tag{5}$$

also requiring $u(\mathbf{r}) = \mathcal{O}(\frac{1}{r^2})$ as $r \to \infty$. The solution can be written

$$u(\mathbf{r}) = \sum_{n=1}^{\infty} \frac{(2n+1)^2}{4\pi n D_n \tau a} (\frac{\tau}{a})^n \Big[(Q_x \cos\varphi + Q_y \sin\varphi) \mathrm{P}_n^1(\cos\theta) + Q_z n \mathrm{P}_n(\cos\theta) \Big] \tag{6}$$

with $D_n = \Big[(n+1)(\sigma_b - \sigma_a)(b/a)^{2n+1} + (n+1)\sigma_a + n\sigma_b \Big]$.

IMAGE SOURCE

Next, an image dipole distribution $\mathbf{R}_{\mathrm{im}}(t)$ in the interval $t \in [0, \tau]$ is determined that produces the potential (6). For the inversion, integral transformations given in [1, 2] are employed. The result is

$$\mathbf{R}_{\mathrm{im}}(t) = \frac{\sigma_b \mathbf{Q}}{(\sigma_a + \sigma_b)} \sum_{k=0}^{\infty} \frac{(\sigma_a - \sigma_b)^k}{(\sigma_a + \sigma_b)^k} (\frac{b}{a})^k \Bigg[4(\frac{b}{a})^{2k} \delta\Big(t - (\frac{b}{a})^{2k}\tau\Big) + \frac{\alpha_k}{\tau} \sqcap \Big(0, t, (\frac{b}{a})^{2k}\tau\Big)$$

$$+\Big(\frac{a^{2k}t}{b^{2k}\tau}\Big)^{-\sigma_a/(\sigma_a+\sigma_b)}\sum_{i=1}^{k+1}\frac{\beta_{k,i}}{\tau(i-1)!}\Big(\ln(\frac{b^{2k}\tau}{a^{2k}t})\Big)^{i-1}\sqcap\Big(0,t,(\frac{b}{a})^{2k}\tau\Big)\Bigg], \tag{7}$$

where

$$\alpha_k=\lim_{s\to 0}\frac{(2s+1)^2(s+1)^k-4s(s+\frac{\sigma_a}{\sigma_a+\sigma_b})^{k+1}}{(s+\frac{\sigma_a}{\sigma_a+\sigma_b})^{k+1}}=\frac{(\sigma_b+\sigma_a)^{k+1}}{\sigma_a^{k+1}}, \tag{8}$$

$$\beta_{k,i}=\lim_{s\to-\frac{\sigma_a}{\sigma_a+\sigma_b}}\frac{\mathrm{d}^{k+1-i}}{\mathrm{d}s^{k+1-i}}\Bigg[\frac{(2s+1)^2(s+1)^k-4s(s+\frac{\sigma_a}{\sigma_a+\sigma_b})^{k+1}}{(k+1-i)!\,s}\Bigg],\qquad \forall i=1,\dots,k{+}1. \tag{9}$$

IMAGE BASED POTENTIAL

The image based potential on the surface $r=a$ may be written:

$$u_{\rm im}(\mathbf{r})\approx\frac{1}{4\pi}\sum_{k=0}^{N}\frac{(\sigma_a-\sigma_b)^k}{(\sigma_a+\sigma_b)^{k+1}}(\frac{b}{a})^k\Bigg[4(\frac{b}{a})^{2k}\frac{\mathbf{Q}\cdot(\mathbf{r}-(\frac{b}{a})^{2k}\tau\hat{\mathbf{z}})}{|\mathbf{r}-(\frac{b}{a})^{2k}\tau\hat{\mathbf{z}}|^3}$$
$$+\frac{\alpha_k\mathbf{Q}\cdot\mathbf{r}}{\tau a^2\sin^2\theta}\Big(\frac{\tau(b/a)^{2k}-a\cos\theta}{|\mathbf{r}-(\frac{b}{a})^{2k}\tau\hat{\mathbf{z}}|}+\cos\theta\Big)-\frac{\alpha_k\mathbf{Q}\cdot\hat{\mathbf{z}}}{\tau a\sin^2\theta}\Big(\frac{\tau(b/a)^{2k}\cos\theta-a}{|\mathbf{r}-(\frac{b}{a})^{2k}\tau\hat{\mathbf{z}}|}+1\Big)$$
$$+\sum_{i=1}^{k+1}\beta_{k,i}\Big(\frac{\sigma_a+\sigma_b}{\sigma_b}\Big)^i(\frac{b}{a})^{2k}\frac{\mathbf{Q}\cdot(\mathbf{r}-(\frac{\sigma_a}{\sigma_a+2\sigma_b})^i(\frac{b}{a})^{2k}\tau\hat{\mathbf{z}})}{|\mathbf{r}-(\frac{\sigma_a}{\sigma_a+2\sigma_b})^i(\frac{b}{a})^{2k}\tau\hat{\mathbf{z}}|^3}\Bigg],$$

where the number of terms of the sum N required for convergence depends on the ratios $\tau:a$, $b:a$, as well as on the relative contrast between the conductivities, *i.e.* of $(\sigma_a-\sigma_b)/(\sigma_a+\sigma_b)$. In effect, the larger the conductivity contrast, the more excentric the source and the thinner the layer, the more terms are needed in the expansion.

The numerical example below illustrates the potential $u_{\rm im}$ and the image $R_{\rm im}$ when $a=1$, $b=0.6$, $\tau=0.5$, and $\sigma_a=1$, $\sigma_b=5$, and the source is $\mathbf{Q}=1\hat{\mathbf{x}}+0\hat{\mathbf{y}}+0\hat{\mathbf{z}}$. Details are discussed in the presentation.

References

[1] J.C.-E. Sten and I.V. Lindell, "Electrostatic image theory for the dielectric sphere with an internal source," *Microw. Opt. Technol. Lett.*, **5**(11), pp. 597-602, 1992.

[2] J.C.-E. Sten and R. Ilmoniemi, "Image theory for a point charge inside a layered dielectric sphere," *Arch. für Electrotech.*, **77**, pp. 327-335, 1994.

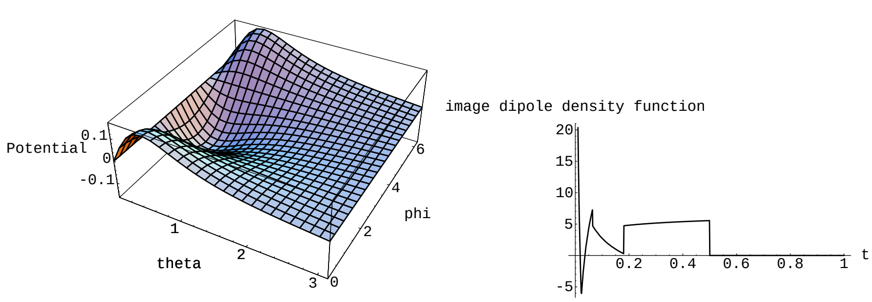

Figure 1: Total image potential and the corresponding continuous part of the image.